Question Sheet 3, Limits III. v1. 2020-21

Divergence

1. i) Let f be defined on a right-hand open interval of $a \in \mathbb{R}$ (i.e. on $(a, a+\eta)$ for some $\eta > 0$). Write out the K - δ definition for

$$\lim_{x \to a+} f(x) = +\infty.$$

Let f be defined on a left-hand open interval of $a \in \mathbb{R}$ (i.e. on $(a-\eta, a)$ for some $\eta > 0$). Write out the K - δ definition for

$$\lim_{x \to a-} f(x) = -\infty.$$

ii) Let f be defined for all sufficiently large positive x. Write out the K - X definitions for each of the following limits,

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to +\infty} f(x) = -\infty,$$

iii) Let f be defined for all sufficiently large negative x. Write out the K - X definitions for each of the following limits.

$$\lim_{x \to -\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty.$$

2. i) Write

$$G(x) = \frac{x}{x^2 - 1}$$

as partial fractions for $x \neq 1$ or -1.

ii) Prove that if $x > 1$ then

$$G(x) > \frac{1}{2(x-1)}.$$

Thus verify the K - δ definition (seen in Question 1i) of

$$\lim_{x \to 1+} G(x) = +\infty.$$

iii) Prove, that if $0 < x < 1$ then

$$G(x) \leq \frac{1}{2\,(x-1)} + \frac{1}{2}.$$

Thus show that the K - δ definition (seen in Question 1i) of

$$\lim_{x \to 1-} G(x) = -\infty$$

is verified by choosing $\delta = \min\left(1, -1/(2K-1)\right)$ for any given $K < 0$.

iv) Evaluate (so there is no need to verify the definition)

$$\lim_{x \to -1+} G(x) \quad \text{and} \quad \lim_{x \to -1-} G(x)\,.$$

v) Evaluate

$$\lim_{x \to +\infty} G(x) \quad \text{and} \quad \lim_{x \to -\infty} G(x)\,,$$

if they exist.

vi) Sketch the graph of G.

3. Verify the K - δ definitions of

$$\text{i)} \lim_{x \to -3} \frac{x^2}{(x+3)^2} = +\infty \quad \text{and} \quad \text{ii)} \lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty.$$

Hint For part i look for a simpler, *lower* bound for $x^2/\left(x+3\right)^2$ while for part ii look for a simpler, *upper* bound for $x/\left(x+3\right)^2$.

4. Define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \frac{1}{x^2+1} + x.$$

Prove by verifying the K - X definitions that

$$\lim_{x \to +\infty} H(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} H(x) = -\infty.$$

Sketch the graph of H.

Limit Rules

5. Using the **Limit Rules** evaluate

   i)
   $$\lim_{x\to 0} \frac{3x^2+4x+1}{x^2+4x+3},$$

   ii)
   $$\lim_{x\to \infty} \frac{3x^2+4x+1}{x^2+4x+3},$$

   iii)
   $$\lim_{x\to -1} \frac{3x^2+4x+1}{x^2+4x+3}.$$

   Note When using a Limit Rule you **must** write down which Rule you are using and you **must** show that any necessary conditions of that rule are satisfied.

6. (i) What is wrong with the argument:

$$\begin{aligned}
\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) &= \lim_{x\to 0} x^3 \times \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \\
&\qquad \text{by the Product Rule} \\
&= 0 \times \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \\
&= 0.
\end{aligned}$$

   (ii) Evaluate
   $$\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right).$$

Exponential and trigonometric examples

7. Recall that in the lectures we have shown that

$$\lim_{x \to 0} e^x = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Use these to evaluate the following limits which include the hyperbolic functions.

(i)

$$\lim_{x \to 0} \frac{\sinh x}{x},$$

ii)

$$\lim_{x \to 0} \frac{\tanh x}{x},$$

iii)

$$\lim_{x \to 0} \frac{\cosh x - 1}{x^2}.$$

8. i) Assuming that $e^x > x$ for all $x > 0$ verify the ε - X definitions of

$$\lim_{x \to +\infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0.$$

Deduce (using the Limit Rules) that

$$\lim_{x \to +\infty} \tanh x = 1 \quad \text{and} \quad \lim_{x \to -\infty} \tanh x = -1.$$

Sketch the graph of $\tanh x$.

Additional Questions

9. i. Prove that

$$\left| e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} \right| < \frac{2}{4!} \left| x^4 \right|$$

for $|x| < 1/2$.

Hint Use the method seen in the notes where it was shown that $|e^x - 1 - x| < |x^2|$ for $|x| < 1/2$.

ii. Deduce

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3} = \frac{1}{6}.$$

iii. Use Part ii. to evaluate

$$\lim_{x\to 0} \frac{\sinh x - x}{x^3}.$$

10. Recall that in the lectures we have shown that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Use this to evaluate (**without** using L'Hôpital's Rule)

i)

$$\lim_{\theta\to 0} \frac{\theta}{\tan\theta},$$

ii)

$$\lim_{\theta\to 0} \frac{\sin\theta - \tan\theta}{\theta^3}.$$